CH5 – CONITNUITY AND DIFFERENTIABLITY

Exercise 5.1 : Solutions of Questions on Page Number : **159**

Q1 : Prove that the function $f(x)=5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

Answer :

The given function is $f(x)=5x-3$

At $x=0$, $f(0)=5\times 0-3=3$

$\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)=5\times 0-3=-3$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, f is continuous at x= 0

At $x=-3$, $f(-3)=5\times(-3)-3=-18$

$\lim_{x\to -3} f(x)=\lim_{x\to -3}(5x-3)=5\times(-3)-3=-18$

$\therefore \lim_{x\to -3} f(x)=f(-3)$

Therefore, f is continuous at x= - 3

At $x=5$, $f(x)=f(5)=5\times 5-3=25-3=22$

$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5\times 5-3=22$

$\therefore \lim_{x\to 5} f(x)=f(5)$

Therefore, f is continuous at x= 5

Answer needs Correction? **Click Here**

Q2 : Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Answer :

The given function is $f(x)=2x^2-1$

At $x=3$, $f(x)=f(3)=2\times 3^2-1=17$

$\lim_{x\to 3} f(x)=\lim_{x\to 3}(2x^2-1)=2\times 3^2-1=17$

$\therefore \lim_{x\to 3} f(x)=f(3)$

Thus, f is continuous at x= 3

Answer needs Correction? **Click Here**

Q3 : Examine the following functions for continuity.

(a) $f(x)=x-5$ (b) $f(x)=\frac{1}{x-5}, x\neq 5$

(c) $f(x)=\frac{x^2-25}{x+5}, x\neq -5$ (d) $f(x)=|x-5|$

Answer :

(a) The given function is $f(x)=x-5$

It is evident that *f* is defined at every real number *k* and its value at *k* is *k* - 5.

It is also observed that, $\lim_{x\to k} f(x) = \lim_{x\to k}(x-5) = k-5 = f(k)$

$\therefore \lim_{x\to k} f(x) = f(k)$

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number *k* ≠ 5, we obtain

$$\lim_{x\to k} f(x) = \lim_{x\to k}\frac{1}{x-5} = \frac{1}{k-5}$$

$$\text{Also, } f(k) = \frac{1}{k-5} \quad (\text{As } k \neq 5)$$

$$\therefore \lim_{x\to k} f(x) = f(k)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(c) The given function is $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

For any real number *c* ≠ - 5, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c}\frac{x^2-25}{x+5} = \lim_{x\to c}\frac{(x+5)(x-5)}{x+5} = \lim_{x\to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x < 5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function *f* is defined at all points of the real line.

Let *c* be a point on a real line. Then, *c* < 5 or *c* = 5 or *c* > 5

Case I: *c* < 5

Then, *f*(*c*) = 5 - *c*

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers less than 5.

Case II : *c* = 5

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}(5-x) = (5-5) = 0$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}(x-5) = 0$$

$$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Therefore, *f* is continuous at *x* = 5

Case III: *c* > 5

Then, $f(c) = f(5) = c - 5$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Answer needs Correction? Click Here

Q4 : Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Answer :

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Then, $\lim_{x \to n} f(n) = \lim_{x \to n} (x^n) = n^n$

$\therefore \lim_{x \to n} f(x) = f(n)$

Therefore, f is continuous at n, where n is a positive integer.

Answer needs Correction? Click Here

Q5 : Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer :

The given function f is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

Then, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5) = 5$

$\therefore \lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$

Therefore, f is not continuous at $x = 1$

Therefore, f is not continuous at $x = 1$

At $x = 2$,

f is defined at 2 and its value at 2 is 5.

Then, $\lim_{x \to 2} f(x) = \lim_{x \to 2} (5) = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$

Answer needs Correction? **Click Here**

Q6 : Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) $c < 2$

(ii) $c > 2$

(iii) $c = 2$

Case (i) $c < 2$

Then, $f(c) = 2c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x+3) = 2c + 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$

Case (ii) $c > 2$

Then, $f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x-3) = 2c - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$

Case (iii) $c = 2$

Then, the left hand limit of f at $x = 2$ is,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x+3) = 2 \times 2 + 3 = 7$

The right hand limit of f at $x = 2$ is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x-3) = 2 \times 2 - 3 = 1$

It is observed that the left and right hand limit of f at $x = 2$ do not coincide.

Therefore, f is not continuous at $x = 2$

Hence, $x = 2$ is the only point of discontinuity of f.

Answer needs Correction? **Click Here**

Q7 : Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}|x|+3, \text{if } x\le -3\\ -2x, \text{if } -3<x<3\\ 6x+2, \text{if } x\ge 3\end{cases}$$

Answer :

The given function f is $f(x)=\begin{cases}|x|+3=-x+3, \text{ if } x\le -3\\ -2x, \text{ if } -3<x<3\\ 6x+2, \text{ if } x\ge 3\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c<-3, \text{ then } f(c)=-c+3$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}(-x+3)=-c+3$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x < -3$

Case II:

$$\text{If } c=-3, \text{ then } f(-3)=-(-3)+3=6$$
$$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}(-x+3)=-(-3)+3=6$$
$$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^+}(-2x)=-2\times(-3)=6$$
$$\therefore \lim_{x\to -3} f(x)=f(-3)$$

Therefore, f is continuous at $x = -3$

Case III:

$$\text{If } -3<c<3, \text{ then } f(c)=-2c \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(-2x)=-2c$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(-2x)=-2\times 3=-6$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(6x+2)=6\times 3+2=20$$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(6x + 2) = 6c + 2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f.

Answer needs Correction? **Click Here**

Q8 : Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points $x < 0$

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(-1) = -1$

The right hand limit of f at $x = 0$ is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(1) = 1$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$

Therefore, f is not continuous at $x = 0$

Case III:

If $c > 0$, then $f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 0$

Hence, $x = 0$ is the only point of discontinuity of f.

Answer needs Correction? **Click Here**

Q9 : Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$\Rightarrow f(x) = -1$ for all $x \in \mathbf{R}$

Let c be any real number. Then, $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Answer needs Correction? **Click Here**

Q10 : Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^2+1$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(x^2+1)=c^2+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<1$

Case II:

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at $x=1$ is,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+1)=1^2+1=2$

The right hand limit of f at $x=1$ is,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x+1)=1+1=2$

$\therefore \lim_{x\to 1} f(x)=f(1)$

Therefore, f is continuous at $x=1$

Case III:

If $c>1$, then $f(c)=c+1$

$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>1$

Hence,the given function f has no point of discontinuity.

Answer needs Correction? **Click Here**

Q11 : Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$$

Answer :

The given function f is $f(x)=\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<2$, then $f(c)=c^3-3$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(x^3-3)=c^3-3$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<2$

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$$
$$\therefore \lim_{x \to 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 2$

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Answer needs Correction? **Click Here**

Q12 : Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^{10} - 1$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (x^{10} - 1) = c^{10} - 1$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2) = 1^2 = 1$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c^2$

If $c > 1$, then $f(c) = c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2) = c^2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Answer needs Correction? **Click Here**

Q13 : Is the function defined by

$$f(x) = \begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Answer :

The given function is $f(x) = \begin{cases} x+5, & \text{if } x \le 1 \\ x-5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (x+5) = c + 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 1$

Case II:

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+5) = 1 + 5 = 6$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-5) = 1 - 5 = -4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $c > 1$, then $f(c) = c - 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (x-5) = c - 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f.

Answer needs Correction? Click Here

Q14 : Discuss the continuity of the function *f*, where *f* is defined by

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Answer :

The given function is $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

The given function is defined at all points of the interval [0, 10].

Let *c* be a point in the interval [0, 10].

Case I:

If $0\le c<1$, then $f(c)=3$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(3)=3$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous in the interval [0, 1).

Case II:

If $c=1$, then $f(3)=3$

The left hand limit of *f* at *x* = 1 is,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3)=3$

Theright hand limit of *f* at *x* = 1 is,

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4)=4$

It is observed that the left and right hand limits of *f* at *x*= 1 do not coincide.

Therefore, *f* is not continuous at *x*= 1

Case III:

If $1<c<3$, then $f(c)=4$ and $\lim_{x\to c} f(x)=\lim_{x\to c}(4)=4$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points of the interval (1, 3).

Case IV:

If $c=3$, then $f(c)=5$

The left hand limit of *f* at *x* = 3 is,

$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(4)=4$

The right hand limit of *f* at *x* = 3 is,

$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(5)=5$

It is observed that the left and right hand limits of *f* at *x*= 3 do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $3 < c \le 10$, then $f(c) = 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$

$\lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ and $x = 3$

Answer needs Correction? **Click Here**

Q15 : Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer :

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x = 0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x) = 2 \times 0 = 0$

The right hand limit of f at $x = 0$ is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$ and $\lim_{x \to c} f(x) = \lim_{x \to c}(0) = 0$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (0, 1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

Theleft hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case V:

If $c < 1$, then $f(c) = 4c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, f is not continuous only at $x = 1$

Answer needs Correction? Click Here

Q16 : Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < -1$

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2 \times (-1) = -2$$

$\therefore \lim_{x \to -1} f(x) = f(-1)$

Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval (- 1, 1).

Case IV:

If $c = 1$, then $f(c) = f(1) = 2 \times 1 = 2$

The left hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$

$\therefore \lim_{x \to 1} f(x) = f(c)$

Therefore, f is continuous at $x = 2$

Case V:

If $c > 1$, then $f(c) = 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (2) = 2$

$\lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 1$

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Answer needs Correction? **Click Here**

Q17 : Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x)=\begin{cases} ax+1, & \text{if } x\le 3 \\ bx+3, & \text{if } x>3 \end{cases}$$

is continuous at $x = 3$.

Answer :

The given function *f* is $f(x)=\begin{cases} ax+1, & \text{if } x\le 3 \\ bx+3, & \text{if } x>3 \end{cases}$

If *f* is continuous at $x = 3$, then

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3) \qquad ...(1)$$

Also,

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(ax+1)=3a+1$$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(bx+3)=3b+3$$

$$f(3)=3a+1$$

Therefore, from (1), we obtain

$$3a+1=3b+3=3a+1$$

$$\Rightarrow 3a+1=3b+3$$

$$\Rightarrow 3a=3b+2$$

$$\Rightarrow a=b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a=b+\frac{2}{3}$

Answer needs Correction? **Click Here**

Q18 : For what value of λ is the function defined by

$$f(x)=\begin{cases} \lambda(x^2-2x), & \text{if } x\le 0 \\ 4x+1, & \text{if } x>0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

Answer :

The given function *f* is $f(x)=\begin{cases} \lambda(x^2-2x), & \text{if } x\le 0 \\ 4x+1, & \text{if } x>0 \end{cases}$

If *f* is continuous at $x = 0$, then

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\Rightarrow \lim_{x\to 0^-}\lambda(x^2-2x)=\lim_{x\to 0^+}(4x+1)=\lambda(0^2-2\times 0)$$

$$\Rightarrow \lambda(0^2-2\times 0)=4\times 0+1=0$$

$$\Rightarrow 0=1=0, \text{ which is not possible}$$

Therefore, there is no value of ÃŽÂ»for which *f* is continuous at $x = 0$

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$\lim_{x \to 1}(4x+1) = 4 \times 1 + 1 = 5$

$\therefore \lim_{x \to 1} f(x) = f(1)$

Therefore, for any values of ÃŽÂ», f is continuous at $x = 1$

Answer needs Correction? Click Here

Q19 : Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point. Here $[x]$ denotes the greatest integer less than or equal to x.

Answer :

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$g(n) = n - [n] = n - n = 0$

The left hand limit of f at $x = n$ is,

$\lim_{x \to n^-} g(x) = \lim_{x \to n^-}(x - [x]) = \lim_{x \to n^-}(x) - \lim_{x \to n^-}[x] = n - (n-1) = 1$

The right hand limit of f at $x = n$ is,

$\lim_{x \to n^+} g(x) = \lim_{x \to n^+}(x - [x]) = \lim_{x \to n^+}(x) - \lim_{x \to n^+}[x] = n - n = 0$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Answer needs Correction? Click Here

Q20 : Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x =$?

Answer :

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x =$.

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x \to \pi} f(x) = \lim_{x \to \pi}(x^2 - \sin x + 5)$

Put $x = \pi + h$

If $x \to \pi$, *then it is evident that* $h \to 0$

$$\begin{aligned}\therefore \lim_{x \to \pi} f(x) &= \lim_{x \to \pi}(x^2 - \sin x + 5) \\ &= \lim_{h \to 0}\left[(\pi + h)^2 - \sin(\pi + h) + 5\right] \\ &= \lim_{h \to 0}(\pi + h)^2 - \lim_{h \to 0}\sin(\pi + h) + \lim_{h \to 0} 5\end{aligned}$$

$$= (\pi+0)^2 - \lim_{h\to 0}[\sin\pi\cosh + \cos\pi\sinh] + 5$$
$$= \pi^2 - \lim_{h\to 0}\sin\pi\cosh - \lim_{h\to 0}\cos\pi\sinh + 5$$
$$= \pi^2 - \sin\pi\cos 0 - \cos\pi\sin 0 + 5$$
$$= \pi^2 - 0\times 1 - (-1)\times 0 + 5$$
$$= \pi^2 + 5$$
$$\therefore \lim_{x\to\pi} f(x) = f(\pi)$$

Therefore, the given function *f* is continuous at $x = \pi$

Answer needs Correction? **Click Here**

Q21 : Discuss the continuity of the following functions.

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \times \cos x$

Answer :

It is known that if *g* and *h* are two continuous functions, then

$g+h,\ g-h,$ and $g.h$ are also continuous.

It has to proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let *c* be a real number. Put $x = c + h$

If x Ã¢â€ ' c, then h Ã¢â€ '0

$$g(c) = \sin c$$
$$\lim_{x\to c} g(x) = \lim_{x\to c}\sin x$$
$$= \lim_{h\to 0}\sin(c+h)$$
$$= \lim_{h\to 0}[\sin c\cos h + \cos c\sin h]$$
$$= \lim_{h\to 0}(\sin c\cos h) + \lim_{h\to 0}(\cos c\sin h)$$
$$= \sin c\cos 0 + \cos c\sin 0$$
$$= \sin c + 0$$
$$= \sin c$$
$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, *g* is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let *c* be a real number. Put $x = c + h$

If x Ã¢â€ ' c, then h Ã¢â€ '0

$h(c) = \cos c$

$$\lim_{x\to c} h(x) = \lim_{x\to c}\cos x$$

$$= \lim_{h\to 0}\cos(c+h)$$
$$= \lim_{h\to 0}[\cos c\cos h - \sin c\sin h]$$
$$= \lim_{h\to 0}\cos c\cos h - \lim_{h\to 0}\sin c\sin h$$
$$= \cos c\cos 0 - \sin c\sin 0$$
$$= \cos c\times 1 - \sin c\times 0$$
$$= \cos c$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function

Answer needs Correction? **Click Here**

Q22 : Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer :

It is known that if g and h are two continuous functions, then

$$(i)\ \frac{h(x)}{g(x)},\ g(x)\neq 0 \text{ is continuous}$$
$$(ii)\ \frac{1}{g(x)},\ g(x)\neq 0 \text{ is continuous}$$
$$(iii)\ \frac{1}{h(x)},\ h(x)\neq 0 \text{ is continuous}$$

It has to be proved first that $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

Let $g(x) = \sin x$

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$
$$\lim_{x\to c} g(x) = \lim_{x\to c}\sin x$$
$$= \lim_{h\to 0}\sin(c+h)$$
$$= \lim_{h\to 0}[\sin c\cos h + \cos c\sin h]$$
$$= \lim_{h\to 0}(\sin c\cos h) + \lim_{h\to 0}(\cos c\sin h)$$
$$= \sin c\cos 0 + \cos c\sin 0$$
$$= \sin c + 0$$
$$= \sin c$$
$$\therefore \lim_{x\to c} g(x) = g(c)$$

Therefore, g is a continuous function.

Let $h(x) = \cos x$

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If x Â® c, then h Â®0

$h(c) = \cos c$

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c}\cos x\\ &= \lim_{h\to 0}\cos(c+h)\\ &= \lim_{h\to 0}[\cos c\cos h - \sin c\sin h]\\ &= \lim_{h\to 0}\cos c\cos h - \lim_{h\to 0}\sin c\sin h\\ &= \cos c\cos 0 - \sin c\sin 0\\ &= \cos c\times 1 - \sin c\times 0\\ &= \cos c\\ \therefore \lim_{x\to c} h(x) &= h(c)\end{aligned}$$

Therefore, $h(x) = \cos x$ is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at $x = n\pi$, n ÃƒÅ½ Z

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

$$\cot x = \frac{\cos x}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \cot x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cotangent is continuous except at $x = n\pi$, n ÃƒÅ½ Z

Answer needs Correction? **Click Here**

Q23 : Find the points of discontinuity of f, where

$$f(x) = \begin{cases}\dfrac{\sin x}{x}, & \text{if } x < 0\\ x+1, & \text{if } x \geq 0\end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases}\dfrac{\sin x}{x}, & \text{if } x < 0\\ x+1, & \text{if } x \geq 0\end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x<0$

Case II:

$$\text{If } c>0, \text{ then } f(c)=c+1 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>0$

Case III:

$$\text{If } c=0, \text{ then } f(c)=f(0)=0+1=1$$

The left hand limit of f at $x=0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1$$

The right hand limit of f at $x=0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$
$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, f is continuous at $x=0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Answer needs Correction? **Click Here**

Q24 : Determine if f defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is a continuous function?

Answer :

The given function f is $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c\neq 0, \text{ then } f(c)=c^2\sin\frac{1}{c}$$
$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c}x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \leq \sin\frac{1}{x} \leq 1,\ x \neq 0$

$$\Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \to 0}\left(-x^2\right) \leq \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = 0$$

Similarly, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Answer needs Correction? **Click Here**

Q25 : Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer :

The given function f is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x \neq 0$

Case II:

If $c = 0$, then $f(0) = -1$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$
$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Answer needs Correction? **Click Here**

Q26 : Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer :

The given function f is $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$ equals the limit of f at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

Put $x = \frac{\pi}{2} + h$

Then, $x \to \frac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$
$$= k \lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2}.1 = \frac{k}{2}$$
$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$
$$\Rightarrow \frac{k}{2} = 3$$
$$\Rightarrow k = 6$$

Therefore, the required value of k is 6.

Answer needs Correction? **Click Here**

Answer needs Correction? Click Here

Q27 : Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases} \qquad \text{at } x = 2$$

Answer :

The given function is $f(x)=\begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function f is continuous at x= 2, if f is defined at x= 2 and if the value of f at x = 2 equals the limit of f at x = 2

It is evident that f is defined at x= 2 and $f(2)=k(2)^2=4k$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(kx^2)=\lim_{x\to 2^+}(3)=4k$$
$$\Rightarrow k\times 2^2=3=4k$$
$$\Rightarrow 4k=3=4k$$
$$\Rightarrow 4k=3$$
$$\Rightarrow k=\frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Answer needs Correction? Click Here

Q28 : Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer :

The given function is $f(x)=\begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at x= p, if f is defined at x= pand if the value of f at x= pequals the limit of f at$x$= p

It is evident that f is defined at$x$= pand $f(\pi)=k\pi+1$

$$\lim_{x\to \pi^-} f(x)=\lim_{x\to \pi^+} f(x)=f(\pi)$$
$$\Rightarrow \lim_{x\to \pi^-}(kx+1)=\lim_{x\to \pi^+}\cos x=k\pi+1$$
$$\Rightarrow k\pi+1=\cos\pi=k\pi+1$$
$$\Rightarrow k\pi+1=-1=k\pi+1$$
$$\Rightarrow k=-\frac{2}{\pi}$$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Answer needs Correction? Click Here

Q29 : Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases} \quad \text{at } x=5$$

Answer :

The given function f is $f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$

The given function f is continuous at x= 5, if f is defined at x= 5 and if the value of f at x = 5 equals the limit of f at x = 5

It is evident that f is defined at x= 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Answer needs Correction? Click Here

Q30 : Find the values of a and b such that the function defined by

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

is a continuous function.

Answer :

The given function f is $f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at x = 2 and x = 10

Since f is continuous at x = 2, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(5) = \lim_{x\to 2^+}(ax+b) = 5$$
$$\Rightarrow 5 = 2a+b = 5$$
$$\Rightarrow 2a+b = 5 \qquad ...(1)$$

Since f is continuous at $x = 10$, we obtain

$$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b) = \lim_{x\to 10^+}(21) = 21$$
$$\Rightarrow 10a+b = 21 = 21$$
$$\Rightarrow 10a+b = 21 \qquad ...(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

Byputting $a = 2$ in equation (1), we obtain

$2 \times 2 + b = 5$

$\Rightarrow 4 + b = 5$

$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Answer needs Correction? **Click Here**

Q31 : Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Answer :

The given function is $f(x) = \cos(x^2)$

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$

$$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$$

It has to be first proved that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, $g(c) = \cos c$

Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\lim_{x\to c} g(x) = \lim_{x\to c}\cos x$$
$$= \lim_{h\to 0}\cos(c+h)$$
$$= \lim_{h\to 0}[\cos c\cos h - \sin c\sin h]$$
$$= \lim_{h\to 0}\cos c\cos h - \lim_{h\to 0}\sin c\sin h$$

$= \cos c \cos 0 - \sin c \sin 0$

$= \cos c \times 1 - \sin c \times 0$

$= \cos c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, $g(x) = \cos x$ is continuous function.

$h(x) = x^2$

Clearly, h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

$\therefore \lim_{x \to k} h(x) = h(k)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Answer needs Correction? **Click Here**

Q32 : Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer :

The given function is $f(x) = |\cos x|$

This function $f$is defined for every real number and $f$can be written as the composition of two functions as,

$f = g$ o h, where $g(x) = |x|$ and $h(x) = \cos x$

$\left[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)\right]$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} x = c$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, *g* is continuous at all points *x*, such that *x*> 0

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(-x) = 0$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(x) = 0$$
$$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+}(x) = g(0)$$

Therefore, *g* is continuous at *x*= 0

From the above three observations, it can be concluded that *g*is continuous at all points.

h (*x*) = cos *x*

It is evident that *h*(*x*) = cos *x* is defined for every real number.

Let *c* be a real number. Put *x*= *c* + *h*

If *x* Ã¢â€ ' *c*, then *h* Ã¢â€ '0

h (*c*) = cos *c*

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c}\cos x\\ &= \lim_{h\to 0}\cos(c+h)\\ &= \lim_{h\to 0}[\cos c\cos h - \sin c\sin h]\\ &= \lim_{h\to 0}\cos c\cos h - \lim_{h\to 0}\sin c\sin h\\ &= \cos c\cos 0 - \sin c\sin 0\\ &= \cos c\times 1 - \sin c\times 0\\ &= \cos c\end{aligned}$$
$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, *h* (*x*) = cos *x* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Answer needs Correction? **Click Here**

Q33 : Examine that $\sin|x|$ is a continuous function.

Answer :

Let $f(x) = \sin|x|$

This function *f*is defined for every real number and *f*can be written as the composition of two functions as,

f= *g o h*, where $g(x) = |x|$ and $h(x) = \sin x$

$$\left[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)\right]$$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

$$\text{If } c<0, \text{ then } g(c)=-c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

$$\text{If } c>0, \text{ then } g(c)=c \text{ and } \lim_{x\to c} g(x)=\lim_{x\to c} x=c$$
$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, *g* is continuous at all points *x*, such that *x*> 0

Case III:

$$\text{If } c=0, \text{ then } g(c)=g(0)=0$$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$$
$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$$
$$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$$

Therefore, *g* is continuous at *x*= 0

From the above three observations, it can be concluded that *g*is continuous at all points.

h (*x*) = sin *x*

It is evident that *h*(*x*) = sin*x* is defined for every real number.

Let *c* be a real number. Put *x*= *c* + *k*

If *x* Ã¢â€ ' *c*, then *k* Ã¢â€ '0

h (*c*) = sin *c*

$$\begin{aligned}
h(c)&=\sin c\\
\lim_{x\to c} h(x)&=\lim_{x\to c}\sin x\\
&=\lim_{k\to 0}\sin(c+k)\\
&=\lim_{k\to 0}[\sin c\cos k+\cos c\sin k]\\
&=\lim_{k\to 0}(\sin c\cos k)+\lim_{h\to 0}(\cos c\sin k)\\
&=\sin c\cos 0+\cos c\sin 0\\
&=\sin c+0\\
&=\sin c\\
\therefore \lim_{x\to c} h(x)&=g(c)
\end{aligned}$$

Therefore, *h* is a continuous function.

It is known that for real valued functions *g* and *h*,such that (*g* o *h*) is defined at *c*, if *g* is continuous at *c* and if *f* is continuous at *g* (*c*), then (*f* o *g*) is continuous at *c*.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\sin x)=|\sin x|$ is a continuous function.

Answer needs Correction? **Click Here**

Q34 : Find all the points of discontinuity of *f* defined by $f(x)=|x|-|x+1|$.

Answer :

The given function is $f(x)=|x|-|x+1|$

Thetwo functions, *g*and *h*, are defined as

$g(x)=|x|$ and $h(x)=|x+1|$

Then, *f* = *g* - *h*

Thecontinuity of *g*and *h* is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, *g* is defined for all real numbers.

Let *c* be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim_{x\to c} g(x)=\lim_{x\to c}(-x)=-c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x* < 0

Case II:

If $c>0$, then $g(c)=c$ and $\lim_{x\to c} g(x)=\lim_{x\to c} x=c$

$\therefore \lim_{x\to c} g(x)=g(c)$

Therefore, *g* is continuous at all points *x*, such that *x*> 0

Case III:

If $c=0$, then $g(c)=g(0)=0$

$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(-x)=0$

$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x)=0$

$\therefore \lim_{x\to 0^-} g(x)=\lim_{x\to 0^+}(x)=g(0)$

Therefore, *g* is continuous at *x*= 0

From the above three observations, it can be concluded that *g*is continuous at all points.

$h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases}-(x+1), & \text{if, } x<-1\\ x+1, & \text{if } x\geq -1\end{cases}$$

Clearly, *h* is defined for every real number.

Let *c* be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim_{x \to c} h(x) = \lim_{x \to c}\left[-(x+1)\right] = -(c+1)$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim_{x \to c} h(x) = \lim_{x \to c}(x+1) = c+1$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-}\left[-(x+1)\right] = -(-1+1) = 0$

$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+}(x+1) = (-1+1) = 0$

$\therefore \lim_{x \to -1^-} h(x) = \lim_{h \to -1^+} h(x) = h(-1)$

Therefore, h is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

Answer needs Correction? **Click Here**

Exercise 5.2 : Solutions of Questions on Page Number : 166

Q1 : Differentiate the functions with respect to *x*.

$\sin\left(x^2+5\right)$

Answer :

Answer needs Correction? Click Here

Q2 : Differentiate the functions with respect to *x*.

$\cos(\sin x)$

Answer :

Let $f(x)=\cos(\sin x), u(x)=\sin x,$ and $v(t)=\cos t$

Then, $(vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Thus, *f* is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x)\cdot\frac{d}{dx}(\sin x)=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$$

Answer needs Correction? Click Here

Q3 : Differentiate the functions with respect to *x*.

$\sin(ax+b)$

Answer :

Let $f(x)=\sin(ax+b),\ u(x)=ax+b,$ and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$

Thus, *f* is a composite function of two functions, *u* and *v*.

Put $t = u(x) = ax + b$

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$

$$\frac{dt}{\cdot}=\frac{d}{\cdot}(ax+b)=\frac{d}{\cdot}(ax)+\frac{d}{\cdot}(b)=a+0=a$$

Hence, by chain rule, we obtain

$$\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin(ax+b)\right]&=\cos(ax+b).\frac{d}{dx}(ax+b)\\&=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]\\&=\cos(ax+b).(a+0)\\&=a\cos(ax+b)\end{aligned}$$

Answer needs Correction? Click Here

Q4 : Differentiate the functions with respect to *x*.

$$\sec\left(\tan\left(\sqrt{x}\right)\right)$$

Answer :

Let $f(x)=\sec\left(\tan\sqrt{x}\right), u(x)=\sqrt{x}, v(t)=\tan t$, and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v\left(u(x)\right)\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Thus, *f* is a composite function of three functions, *u*, *v*, and *w*.

Put $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

$$\begin{aligned}\text{Then, }\frac{dw}{ds}&=\frac{d}{ds}(\sec s)=\sec s\tan s=\sec(\tan t).\tan(\tan t) && [s=\tan t]\\&=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right) && \left[t=\sqrt{x}\right]\end{aligned}$$

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}\cdot x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we obtain

$$\begin{aligned}\frac{dt}{dx}&=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}\\&=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}\\&=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)\\&=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}\end{aligned}$$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]&=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)\\&=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)\end{aligned}$$

$$= \sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$
$$= \frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Answer needs Correction? Click Here

Q5 : Differentiate the functions with respect to *x*.

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Answer :

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where $g(x) = \sin(ax + b)$ and

$h(x) = \cos(cx + d)$

$$\therefore f' = \frac{g'h - gh'}{h^2}$$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore$ *g* is a composite function of two functions, *u* and *v*.

Put $t=u(x)=ax+b$

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$
$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Therefore, by chain rule, we obtain

$$g'=\frac{dg}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos(ax+b)\cdot a=a\cos(ax+b)$$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d,\ q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

$\therefore$ *h* is a composite function of two functions, *p* and *q*.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$$
$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c$$

Therefore, by chain rule, we obtain

$$h'=\frac{dh}{dx}=\frac{dq}{dy}\cdot\frac{dy}{dx}=-\sin(cx+d)\times c=-c\sin(cx+d)$$

$$\therefore f'=\frac{a\cos(ax+b)\cdot\cos(cx+d)-\sin(ax+b)\{-c\sin(cx+d)\}}{\left[\cos(cx+d)\right]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)}$$
$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Answer needs Correction? Click Here

Q6 : Differentiate the functions with respect to *x*.

$\cos x^3 . \sin^2(x^5)$

Answer :

The given function is $\cos x^3 . \sin^2(x^5)$.

$$\frac{d}{dx}\left[\cos x^3 \cdot \sin^2(x^5)\right] = \sin^2(x^5) \times \frac{d}{dx}(\cos x^3) + \cos x^3 \times \frac{d}{dx}\left[\sin^2(x^5)\right]$$
$$= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx}(x^3) + \cos x^3 \times 2\sin(x^5) \cdot \frac{d}{dx}\left[\sin x^5\right]$$
$$= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2\sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}(x^5)$$
$$= -3x^2 \sin x^3 \cdot \sin^2(x^5) + 2\sin x^5 \cos x^5 \cos x^3 \cdot \times 5x^4$$
$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)$$

Answer needs Correction? Click Here

Q7 : Differentiate the functions with respect to *x*.

$2\sqrt{\cot(x^2)}$

Answer :

$$\frac{d}{dx}\left[2\sqrt{\cot(x^2)}\right]$$
$$= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx}\left[\cot(x^2)\right]$$
$$= \sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times -\text{cosec}^2(x^2) \times \frac{d}{dx}(x^2)$$
$$= -\sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times \frac{1}{\sin^2(x^2)} \times (2x)$$
$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\sin x^2}$$
$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2 \cos x^2}\sin x^2}$$
$$= \frac{-2\sqrt{2}\,x}{\sin x^2 \sqrt{\sin 2x^2}}$$

Answer needs Correction? Click Here

Q8 : Differentiate the functions with respect to *x*.

$\cos\left(\sqrt{x}\right)$

Answer :

$$\begin{aligned}
&\text{Let } f(x) = \cos\left(\sqrt{x}\right)\\
&\text{Also, let } u(x) = \sqrt{x}\\
&\text{And, } v(t) = \cos t\\
&\text{Then, } (vou)(x) = v(u(x))\\
&\qquad = v\left(\sqrt{x}\right)\\
&\qquad = \cos\sqrt{x}\\
&\qquad = f(x)
\end{aligned}$$

Clearly, *f* is a composite function of two functions, *u* and *v*, such that

$t = u(x) = \sqrt{x}$

$$\begin{aligned}
&\text{Then, } \frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}\\
&\qquad = \frac{1}{2\sqrt{x}}\\
&\text{And, } \frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t\\
&\qquad = -\sin\left(\sqrt{x}\right)
\end{aligned}$$

By using chain rule, we obtain

$$\begin{aligned}
&\frac{dt}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}\\
&= -\sin\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}\\
&= -\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right)\\
&= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}
\end{aligned}$$

Alternate method

$$\begin{aligned}
\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] &= -\sin\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)\\
&= -\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)\\
&= -\sin\sqrt{x}\times\frac{1}{2}x^{-\frac{1}{2}}\\
&= \frac{-\sin\sqrt{x}}{2\sqrt{x}}
\end{aligned}$$

Answer needs Correction? **Click Here**

Q9 : Prove that the function *f* given by

$f(x)=|x-1|,\ x\in\mathbf{R}$ is notdifferentiable at $x = 1$.

Answer :

The given function is $f(x)=|x-1|,\ x\in\mathbf{R}$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$,

consider the left hand limit of f at $x = 1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{|1+h-1|-|1-1|}{h}$$
$$=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h}\qquad (h<0\Rightarrow |h|=-h)$$
$$=-1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{|1+h-1|-|1-1|}{h}$$
$$=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h}\qquad (h>0\Rightarrow |h|=h)$$
$$=1$$

Since the left and right hand limits of f at $x = 1$ are not equal, f is not differentiable at $x = 1$

Answer needs Correction? **Click Here**

Q10 : Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at $x = 1$ and $x = 2$.

Answer :

The given function f is $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x = 1$, consider the left hand limit of f at $x = 1$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$
$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$
$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x = 1$ are not equal, f is not differentiable at

$x = 1$

To check the differentiability of the given function at $x = 2$, consider the left hand limit of f at $x = 2$

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-}\frac{[2+h]-[2]}{h}$$
$$= \lim_{h\to 0^-}\frac{1-2}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

Consider the right hand limit of f at $x = 1$

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+}\frac{[2+h]-[2]}{h}$$
$$= \lim_{h\to 0^+}\frac{2-2}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = 2$ are not equal, f is not differentiable at $x = 2$

Answer needs Correction? **Click Here**

<< Previous Chapter 4 : Determinants

Next Chapter 6 : Application of Derivatives >>

Exercise 5.3 : Solutions of Questions on Page Number : **169**

Q1 : Find $\frac{dy}{dx}$:

$2x + 3y = \sin x$

Answer :

The given relationship is $2x + 3y = \sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x+3y) = \frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$$
$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$
$$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$$
$$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Answer needs Correction? **Click Here**

Q2 : Find $\frac{dy}{dx}$:

$2x + 3y = \sin y$

Answer :

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Answer needs Correction? Click Here

Q3 : Find $\frac{dy}{dx}$:

$ax+by^2=\cos y$

Answer :

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad ...(1)$$

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ $\qquad ...(2)$

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Answer needs Correction? Click Here

Q4 : Find $\frac{dy}{dx}$:

$xy+y^2=\tan x+y$

Answer :

The given relationship is $xy+y^2=\tan x+y$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(xy+y^2)=\frac{d}{dx}(\tan x+y)$$
$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$
$$\Rightarrow \left[y\cdot\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$
$$\Rightarrow y\cdot 1+x.\frac{dy}{dx}+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$
$$\Rightarrow (x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$
$$\therefore \frac{dy}{dx}=\frac{\sec^2 x-y}{(x+2y-1)}$$

Answer needs Correction? **Click Here**

Q5 : Find $\frac{dy}{dx}$:

$x^2+xy+y^2=100$

Answer :

The given relationship is $x^2+xy+y^2=100$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x^2+xy+y^2)=\frac{d}{dx}(100)$$
$$\Rightarrow \frac{d}{dx}(x^2)+\frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=0 \text{ [Derivative of constant function is 0]}$$
$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad \text{[Using product rule and chain rule]}$$
$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$
$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$
$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Answer needs Correction? **Click Here**

Q6 : Find $\frac{dy}{dx}$:

$x^3+x^2y+xy^2+y^3=81$

Answer :

The given relationship is $x^3+x^2y+xy^2+y^3=81$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x^3+x^2y+xy^2+y^3)=\frac{d}{dx}(81)$$
$$\Rightarrow \frac{d}{dx}(x^3)+\frac{d}{dx}(x^2y)+\frac{d}{dx}(xy^2)+\frac{d}{dx}(y^3)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y\cdot 2x+x^2\frac{dy}{dx}\right]+\left[y^2\cdot 1+x\cdot 2y\cdot\frac{dy}{dx}\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Answer needs Correction? Click Here

Q7 : Find $\frac{dy}{dx}$:

$\sin^2 y+\cos xy=\pi$

Answer :

The given relationship is $\sin^2 y+\cos xy=\pi$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 y+\cos xy\right)=\frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right)+\frac{d}{dx}(\cos xy)=0 \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right)=2\sin y\frac{d}{dx}(\sin y)=2\sin y\cos y\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}(\cos xy)=-\sin xy\frac{d}{dx}(xy)=-\sin xy\left[y\frac{d}{dx}(x)+x\frac{dy}{dx}\right]$$

$$=-\sin xy\left[y.1+x\frac{dy}{dx}\right]=-y\sin xy-x\sin xy\frac{dy}{dx} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx}-y\sin xy-x\sin xy\frac{dy}{dx}=0$$

$$\Rightarrow (2\sin y\cos y-x\sin xy)\frac{dy}{dx}=y\sin xy$$

$$\Rightarrow (\sin 2y-x\sin xy)\frac{dy}{dx}=y\sin xy$$

$$\therefore \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

Answer needs Correction? Click Here

Q8 : Find $\frac{dy}{dx}$:

$\sin^2 x+\cos^2 y=1$

Answer :

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}\left(\sin^2 x + \cos^2 y\right) = \frac{d}{dx}(1)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right) + \frac{d}{dx}\left(\cos^2 y\right) = 0$$
$$\Rightarrow 2\sin x \cdot \frac{d}{dx}(\sin x) + 2\cos y \cdot \frac{d}{dx}(\cos y) = 0$$
$$\Rightarrow 2\sin x \cos x + 2\cos y(-\sin y) \cdot \frac{dy}{dx} = 0$$
$$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Answer needs Correction? **Click Here**

Q9 : Find $\frac{dy}{dx}$:

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer :

The given relationship is $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$
$$\Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad ...(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{\left(1+x^2\right).\frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$
$$= \frac{\left(1+x^2\right).2 - 2x \cdot [0+2x]}{\left(1+x^2\right)^2} = \frac{2+2x^2-4x^2}{\left(1+x^2\right)^2} = \frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2} \qquad ...(2)$$

Also, $\sin y = \frac{2x}{1+x^2}$

$$\Rightarrow \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}$$

$$= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} = \frac{1-x^2}{1+x^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Answer needs Correction? Click Here

Q10 : Find $\frac{dy}{dx}$:

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Answer :

The given relationship is $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y = \frac{3x-x^3}{1-3x^2} \qquad ...(1)$$

It is known that, $\tan y = \frac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ ...(2)

Comparing equations (1) and (2), we obtain

$$x = \tan\frac{y}{3}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3} \cdot \frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1 = \sec^2\frac{y}{3} \cdot \frac{1}{3} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$$

Answer needs Correction? Click Here

Q11 : Find $\frac{dy}{dx}$:

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$$

Answer :

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to *x*, we obtain

Answer needs Correction? **Click Here**

Q12 : Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ 0<x<1$$

Answer :

The given relationship is $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y)=\cos y\cdot\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$= \sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx} \qquad ...(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{(1+x^2)\cdot(1-x^2)' - (1-x^2)\cdot(1+x^2)'}{(1+x^2)^2} \qquad \text{[Using quotient rule]}$$

$$= \frac{(1+x^2)(-2x)-(1-x^2)\cdot(2x)}{(1+x^2)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Alternate method

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow (1+x^2)\sin y = 1-x^2$$

$$\Rightarrow (1+\sin y)x^2 = 1-\sin y$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x = \frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x = \frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x = \tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\cdot\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right) \cdot \frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

$$\Rightarrow 1 = \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\right] \cdot \left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow 1 = \left(1 + x^2\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1 + x^2}$$

Answer needs Correction? **Click Here**

Q13 : Find $\frac{dy}{dx}$:

$$y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right), -1 < x < 1$$

Answer :

The given relationship is $y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$

$$y = \cos^{-1}\left(\frac{2x}{1 + x^2}\right)$$

$$\Rightarrow \cos y = \frac{2x}{1 + x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(\cos y) = \frac{d}{dx} \cdot \left(\frac{2x}{1 + x^2}\right)$$

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = \frac{\left(1 + x^2\right) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}\left(1 + x^2\right)}{\left(1 + x^2\right)^2}$$

$$\Rightarrow -\sqrt{1 - \cos^2 y}\,\frac{dy}{dx} = \frac{\left(1 + x^2\right) \times 2 - 2x \cdot 2x}{\left(1 + x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1 - \left(\frac{2x}{1 + x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2\left(1 - x^2\right)}{\left(1 + x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1 + x^2\right)^2 - 4x^2}{\left(1 + x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1 - x^2\right)}{\left(1 + x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1 - x^2\right)^2}{\left(1 + x^2\right)^2}}\,\frac{dy}{dx} = \frac{-2\left(1 - x^2\right)}{\left(1 + x^2\right)^2}$$

$$\Rightarrow \frac{1 - x^2}{1 + x^2} \cdot \frac{dy}{dx} = \frac{-2\left(1 - x^2\right)}{\left(1 + x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1 + x^2}$$

Answer needs Correction? **Click Here**

Q14 : Find $\frac{dy}{dx}$:

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$$

Answer :

The given relationship is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$
$$\Rightarrow \sin y=2x\sqrt{1-x^2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$
$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$
$$\Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$
$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Answer needs Correction? Click Here

Q15 : Find $\frac{dy}{dx}$:

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right),0<x<\frac{1}{\sqrt{2}}$$

Answer :

The given relationship is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$
$$\Rightarrow \sec y=\frac{1}{2x^2-1}$$
$$\Rightarrow \cos y=2x^2-1$$
$$\Rightarrow 2x^2=1+\cos y$$
$$\Rightarrow 2x^2=2\cos^2\frac{y}{}$$

$$\Rightarrow x = \cos\frac{y}{2}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin\frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin\frac{y}{2}} = \frac{-2}{\sqrt{1-\cos^2\frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

Answer needs Correction? **Click Here**

Exercise 5.4 : Solutions of Questions on Page Number : 174

Q1 : Differentiate the following w.r.t. *x*:

$$\frac{e^x}{\sin x}$$

Answer :

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{\sin x.(e^x) - e^x \cdot (\cos x)}{\sin^2 x}$$
$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$$

Answer needs Correction? Click Here

Q2 : Differentiate the following w.r.t. *x*:

$$e^{\sin^{-1} x}$$

Answer :

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$
$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}\left(\sin^{-1} x\right)$$
$$= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$
$$= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$
$$\therefore \frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1,1)$$

Answer needs Correction? Click Here

Q3 : Differentiate the following w.r.t. *x*:

$$e^{x^3}$$

Answer :

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{x^3}\right)=e^{x^3}\cdot\frac{d}{dx}\left(x^3\right)=e^{x^3}\cdot 3x^2=3x^2e^{x^3}$$

Answer needs Correction? Click Here

Q4 : Differentiate the following w.r.t. *x*:

$\sin\left(\tan^{-1}e^{-x}\right)$

Answer :

Let $y=\sin\left(\tan^{-1}e^{-x}\right)$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left[\sin\left(\tan^{-1}e^{-x}\right)\right]\\&=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{d}{dx}\left(\tan^{-1}e^{-x}\right)\\&=\cos\left(\tan^{-1}e^{-x}\right)\cdot\frac{1}{1+\left(e^{-x}\right)^2}\cdot\frac{d}{dx}\left(e^{-x}\right)\\&=\frac{\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\cdot e^{-x}\cdot\frac{d}{dx}(-x)\\&=\frac{e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\times(-1)\\&=\frac{-e^{-x}\cos\left(\tan^{-1}e^{-x}\right)}{1+e^{-2x}}\end{aligned}$$

Answer needs Correction? Click Here

Q5 : Differentiate the following w.r.t. *x*:

$\log\left(\cos e^x\right)$

Answer :

Let $y=\log\left(\cos e^x\right)$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left[\log\left(\cos e^x\right)\right]\\&=\frac{1}{\cos e^x}\cdot\frac{d}{dx}\left(\cos e^x\right)\\&=\frac{1}{\cos e^x}\cdot\left(-\sin e^x\right)\cdot\frac{d}{dx}\left(e^x\right)\\&=\frac{-\sin e^x}{\cos e^x}\cdot e^x\\&=-e^x\tan e^x, e^x\neq(2n+1)\frac{\pi}{2}, n\in\mathbf{N}\end{aligned}$$

Answer needs Correction? Click Here

Q6 : Differentiate the following w.r.t. *x*:

$e^x + e^{x^2} + ... + e^{x^5}$

Answer :

$$\frac{d}{dx}\left(e^x + e^{x^2} + ... + e^{x^5}\right)$$
$$= \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$$
$$= e^x + \left[e^{x^2} \times \frac{d}{dx}\left(x^2\right)\right] + \left[e^{x^3} \cdot \frac{d}{dx}\left(x^3\right)\right] + \left[e^{x^4} \cdot \frac{d}{dx}\left(x^4\right)\right] + \left[e^{x^5} \cdot \frac{d}{dx}\left(x^5\right)\right]$$
$$= e^x + \left(e^{x^2} \times 2x\right) + \left(e^{x^3} \times 3x^2\right) + \left(e^{x^4} \times 4x^3\right) + \left(e^{x^5} \times 5x^4\right)$$
$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Answer needs Correction? Click Here

Q7 : Differentiate the following w.r.t. *x*:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer :

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

By differentiating this relationship with respect to *x*, we obtain

$$y^2 = e^{\sqrt{x}}$$
$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$
$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, \ x > 0$$

Answer needs Correction? Click Here

Q8 : Differentiate the following w.r.t. *x*:

$\log(\log x), x > 1$

Answer :

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]$$
$$=\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$
$$=\frac{1}{\log x}\cdot\frac{1}{x}$$
$$=\frac{1}{x\log x},\ x>1$$

Answer needs Correction? Click Here

Q9 : Differentiate the following w.r.t. *x*:

$$\frac{\cos x}{\log x}, x>0$$

Answer :

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$
$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$
$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2}, x>0$$

Answer needs Correction? Click Here

Q10 : Differentiate the following w.r.t. *x*:

$$\cos\left(\log x+e^x\right), x>0$$

Answer :

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we obtain

$$\frac{dy}{dx}=-\sin\left(\log x+e^x\right)\cdot\frac{d}{dx}\left(\log x+e^x\right)$$
$$=-\sin\left(\log x+e^x\right)\cdot\left[\frac{d}{dx}(\log x)+\frac{d}{dx}\left(e^x\right)\right]$$
$$=-\sin\left(\log x+e^x\right)\cdot\left(\frac{1}{x}+e^x\right)$$
$$=-\left(\frac{1}{x}+e^x\right)\sin\left(\log x+e^x\right), x>0$$

Exercise 5.5 : Solutions of Questions on Page Number : **178**

Q1 : Differentiate the function with respect to *x*.

$\cos x.\cos 2x.\cos 3x$

Answer :

Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$\log y = \log(\cos x.\cos 2x.\cos 3x)$

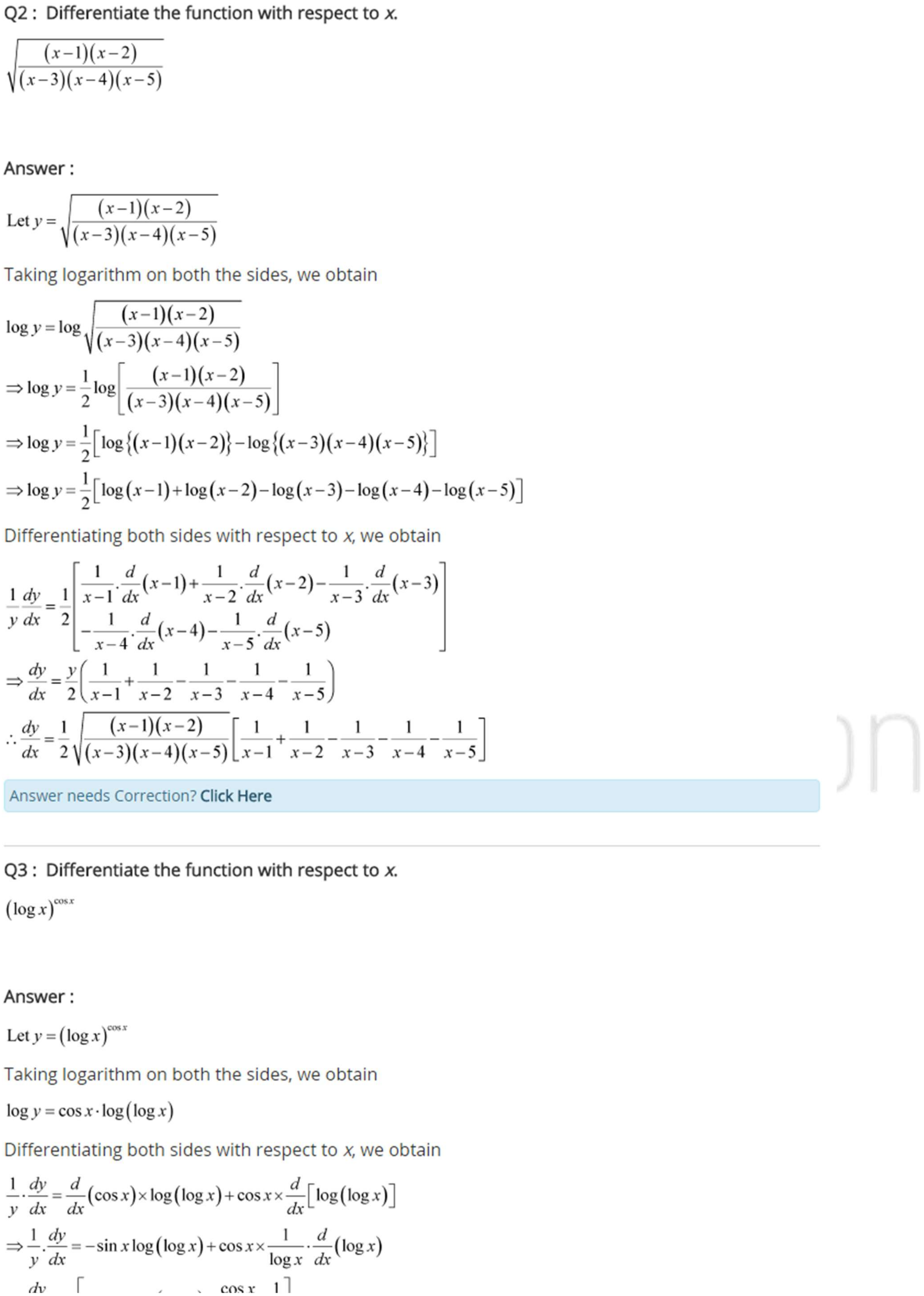
$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x) + \frac{1}{\cos 2x}\cdot\frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x}\cdot\frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}(3x)\right]$$

$$\therefore \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x[\tan x + 2\tan 2x + 3\tan 3x]$$



Q2 : Differentiate the function with respect to *x*.

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Answer :

$$\text{Let } y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}.\frac{d}{dx}(x-1)+\frac{1}{x-2}.\frac{d}{dx}(x-2)-\frac{1}{x-3}.\frac{d}{dx}(x-3)\\-\frac{1}{x-4}.\frac{d}{dx}(x-4)-\frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Answer needs Correction? **Click Here**

Q3 : Differentiate the function with respect to *x*.

$$(\log x)^{\cos x}$$

Answer :

$$\text{Let } y=(\log x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\log y=\cos x\cdot\log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right]$$

Answer needs Correction? **Click Here**

Q4 : Differentiate the function with respect to *x*.

$x^x - 2^{\sin x}$

Answer :

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$\therefore y = u - v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$u = x^x$

Taking logarithm on both the sides, we obtain

$\log u = x \log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$v = 2^{\sin x}$

Taking logarithm on both the sides with respect to *x*, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$$

Answer needs Correction? **Click Here**

Q5 : Differentiate the function with respect to *x*.

$(x+3)^2.(x+4)^3.(x+5)^4$

Answer :

Let $y=(x+3)^2.(x+4)^3.(x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y=\log(x+3)^2+\log(x+4)^3+\log(x+5)^4$$
$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=2\cdot\frac{1}{x+3}\cdot\frac{d}{dx}(x+3)+3\cdot\frac{1}{x+4}\cdot\frac{d}{dx}(x+4)+4\cdot\frac{1}{x+5}\cdot\frac{d}{dx}(x+5)$$
$$\Rightarrow \frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\cdot\left[2(x^2+9x+20)+3(x^2+8x+15)+4(x^2+7x+12)\right]$$
$$\therefore \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Answer needs Correction? **Click Here**

Q6 : Differentiate the function with respect to *x*.

$(x+3)^2.(x+4)^3.(x+5)^4$

Answer :

Let $y=(x+3)^2.(x+4)^3.(x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y=\log(x+3)^2+\log(x+4)^3+\log(x+5)^4$$
$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=2\cdot\frac{1}{x+3}\cdot\frac{d}{dx}(x+3)+3\cdot\frac{1}{x+4}\cdot\frac{d}{dx}(x+4)+4\cdot\frac{1}{x+5}\cdot\frac{d}{dx}(x+5)$$
$$\Rightarrow \frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\cdot\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3\cdot\left[2(x^2+9x+20)+3(x^2+8x+15)+4(x^2+7x+12)\right]$$

$$\therefore \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Answer needs Correction? Click Here

Q7 : Differentiate the function with respect to *x*.

$(\log x)^x + x^{\log x}$

Answer :

Let $y = (\log x)^x + x^{\log x}$

Also, let $u = (\log x)^x$ and $v = x^{\log x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = (\log x)^x$

$$\Rightarrow \log u = \log\left[(\log x)^x\right]$$

$$\Rightarrow \log u = x\log(\log x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\times\log(\log x) + x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1\times\log(\log x) + x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\frac{\log(\log x)\cdot\log x + 1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1}\left[1 + \log x.\log(\log x)\right] \qquad ...(2)$$

$v = x^{\log x}$

$$\Rightarrow \log v = \log\left(x^{\log x}\right)$$

$$\Rightarrow \log v = \log x\log x = (\log x)^2$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = 2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx} = 2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x - 1} \cdot \log x \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x.\log(\log x)\right] + 2x^{\log x - 1} \cdot \log x$$

Answer needs Correction? Click Here

Q8 : Differentiate the function with respect to *x*.

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer :

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\sin x) + x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \cdot \log(\sin x) + x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x} \cdot \cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \qquad ...(2)$$

$v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to *x*, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (\sin x)^x(x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$$

Answer needs Correction? Click Here

Q9 : Differentiate the function with respect to *x*.

$x^{\sin x}+(\sin x)^{\cos x}$

Answer :

Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Also, let $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{\sin x}$

$\Rightarrow \log u=\log\left(x^{\sin x}\right)$

$\Rightarrow \log u=\sin x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\cdot\log x+\sin x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \qquad ...(2)$$

$v=(\sin x)^{\cos x}$

$\Rightarrow \log v=\log(\sin x)^{\cos x}$

$\Rightarrow \log v=\cos x\log(\sin x)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dv}{dx}=v\left[-\sin x.\log(\sin x)+\cos x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}[-\sin x\log\sin x+\cot x\cos x]$$

$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}[\cot x\cos x-\sin x\log\sin x] \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}[\cos x\cot x-\sin x\log\sin x]$$

Answer needs Correction? **Click Here**

Q10 : Differentiate the function with respect to *x*.

$x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Answer :

Let $y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Also, let $u = x^{x\cos x}$ and $v = \frac{x^2+1}{x^2-1}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

$u = x^{x\cos x}$

$\Rightarrow \log u = \log\left(x^{x\cos x}\right)$

$\Rightarrow \log u = x\cos x \log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\cdot\cos x\cdot\log x + x\cdot\frac{d}{dx}(\cos x)\cdot\log x + x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[1\cdot\cos x\cdot\log x + x\cdot(-\sin x)\log x + x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}(\cos x\log x - x\sin x\log x + \cos x)$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}\left[\cos x(1+\log x) - x\sin x\log x\right] \quad ...(2)$$

$v = \frac{x^2+1}{x^2-1}$

$\Rightarrow \log v = \log\left(x^2+1\right) - \log\left(x^2-1\right)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{2x}{x^2+1} - \frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2x\left(x^2-1\right) - 2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{\left(x^2-1\right)^2} \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{x\cos x}\left[\cos x(1+\log x) - x\sin x\log x\right] - \frac{4x}{\left(x^2-1\right)^2}$$

Answer needs Correction? **Click Here**

Q11 : Differentiate the function with respect to *x*.

$(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Answer :

Let $y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Also, let $u=(x\cos x)^x$ and $v=(x\sin x)^{\frac{1}{x}}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=(x\cos x)^x$

$\Rightarrow \log u=\log(x\cos x)^x$

$\Rightarrow \log u=x\log(x\cos x)$

$\Rightarrow \log u=x[\log x+\log\cos x]$

$\Rightarrow \log u=x\log x+x\log\cos x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log x)+\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)\right\}+\left\{\log\cos x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[\left(\log x\cdot 1+x\cdot\frac{1}{x}\right)+\left\{\log\cos x\cdot 1+x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[(\log x+1)+\left\{\log\cos x+\frac{x}{\cos x}\cdot(-\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x[(1+\log x)+(\log\cos x-x\tan x)]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x[1-x\tan x+(\log x+\log\cos x)]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x[1-x\tan x+\log(x\cos x)] \qquad ...(2)$$

$v=(x\sin x)^{\frac{1}{x}}$

$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}$

$\Rightarrow \log v=\frac{1}{x}\log(x\sin x)$

$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$

$\Rightarrow \log v=\frac{1}{x}\log x+\frac{1}{x}\log\sin x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{x}\log x\right)+\frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}(\log x)\right]+\left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{x}\right]+\left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(1-\log x)+\left[-\frac{\log(\sin x)}{x^2}+\frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}+\frac{-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right] \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = (x\cos x)^x\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x+1-\log(x\sin x)}{x^2}\right]$$

Answer needs Correction? **Click Here**

Q12 : Find $\frac{dy}{dx}$ of function.

$x^y + y^x = 1$

Answer :

The given function is $x^y + y^x = 1$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u + v = 1$

$$\therefore \frac{du}{dx}+\frac{dv}{dx}=0 \quad ...(1)$$

$$u = x^y$$
$$\Rightarrow \log u = \log(x^y)$$
$$\Rightarrow \log u = y\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x\frac{dy}{dx}+y\cdot\frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx} = u\left[\log x\frac{dy}{dx}+y\cdot\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right) \quad ...(2)$$

$$v = y^x$$
$$\Rightarrow \log v = \log(y^x)$$
$$\Rightarrow \log v = x\log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)$$
$$\Rightarrow \frac{dv}{dx} = v\left(\log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right) \quad ...(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow \left(x^y \log x + xy^{x-1}\right)\frac{dy}{dx} = -\left(yx^{y-1} + y^x \log y\right)$$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Answer needs Correction? **Click Here**

Q13 : Find $\frac{dy}{dx}$ of function.

$y^x = x^y$

Answer :

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$x \log y = y \log x$

Differentiating both sides with respect to *x*, we obtain

$$\log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y}\frac{dy}{dx} = \log x \frac{dy}{dx} + \frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y\log x}{y}\right)\frac{dy}{dx} = \frac{y - x\log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

Answer needs Correction? **Click Here**

Q14 : Find $\frac{dy}{dx}$ of function.

$(\cos x)^y = (\cos y)^x$

Answer :

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$y \log \cos x = x \log \cos y$

Differentiating both sides, we obtain

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log\cos x\frac{dy}{dx}+y\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x)=\log\cos y\cdot 1+x\cdot\frac{1}{\cos y}\cdot\frac{d}{dx}(\cos y)$$

$$\Rightarrow \log\cos x\frac{dy}{dx}+\frac{y}{\cos x}\cdot(-\sin x)=\log\cos y+\frac{x}{\cos y}(-\sin y)\cdot\frac{dy}{dx}$$

$$\Rightarrow \log\cos x\frac{dy}{dx}-y\tan x=\log\cos y-x\tan y\frac{dy}{dx}$$

$$\Rightarrow (\log\cos x+x\tan y)\frac{dy}{dx}=y\tan x+\log\cos y$$

$$\therefore \frac{dy}{dx}=\frac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$$

Answer needs Correction? **Click Here**

Q15 : Find $\frac{dy}{dx}$ of function.

$xy=e^{(x-y)}$

Answer :

The given function is $xy=e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy)=\log\left(e^{x-y}\right)$$
$$\Rightarrow \log x+\log y=(x-y)\log e$$
$$\Rightarrow \log x+\log y=(x-y)\times 1$$
$$\Rightarrow \log x+\log y=x-y$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$
$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$
$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$
$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$
$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Answer needs Correction? **Click Here**

Q16 : Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Answer :

The given relationship is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow\frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot 4x^3+\frac{1}{1+x^8}\cdot 8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times 1}{1+1^2}+\frac{4\times 1^3}{1+1^4}+\frac{8\times 1^7}{1+1^8}\right]$

$$=2\times 2\times 2\times 2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Answer needs Correction? Click Here

Q17 : Differentiate $(x^5-5x+8)(x^3+7x+9)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

Answer :

Let $y=(x^5-5x+8)(x^3+7x+9)$

(i)

Let $x^2-5x+8=u$ and $x^3+7x+9=v$

$\therefore y=uv$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}\cdot v+u\cdot\frac{dv}{dx}\qquad\text{(By using product rule)}$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}(x^2-5x+8)\cdot(x^3+7x+9)+(x^2-5x+8)\cdot\frac{d}{dx}(x^3+7x+9)$$

$$\Rightarrow\frac{dy}{dx}=(2x-5)(x^3+7x+9)+(x^2-5x+8)(3x^2+7)$$

$$\Rightarrow\frac{dy}{dx}=2x(x^3+7x+9)-5(x^3+7x+9)+x^2(3x^2+7)-5x(3x^2+7)+8(3x^2+7)$$

$$\Rightarrow\frac{dy}{dx}=(2x^4+14x^2+18x)-5x^3-35x-45+(3x^4+7x^2)-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii)

$$\begin{aligned} y &= \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right) \\ &= x^2\left(x^3 + 7x + 9\right) - 5x\left(x^3 + 7x + 9\right) + 8\left(x^3 + 7x + 9\right) \\ &= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72 \\ &= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72 \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}\left(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72\right) \\ &= \frac{d}{dx}\left(x^5\right) - 5\frac{d}{dx}\left(x^4\right) + 15\frac{d}{dx}\left(x^3\right) - 26\frac{d}{dx}\left(x^2\right) + 11\frac{d}{dx}(x) + \frac{d}{dx}(72) \\ &= 5x^4 - 5\times 4x^3 + 15\times 3x^2 - 26\times 2x + 11\times 1 + 0 \\ &= 5x^4 - 20x^3 + 45x^2 - 52x + 11 \end{aligned}$$

(iii) $y = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left(x^2 - 5x + 8\right) + \log\left(x^3 + 7x + 9\right)$$

Differentiating both sides with respect to *x*, we obtain

$$\begin{aligned} &\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log\left(x^2 - 5x + 8\right) + \frac{d}{dx}\log\left(x^3 + 7x + 9\right) \\ &\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}\cdot\frac{d}{dx}\left(x^2 - 5x + 8\right) + \frac{1}{x^3 + 7x + 9}\cdot\frac{d}{dx}\left(x^3 + 7x + 9\right) \\ &\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2 - 5x + 8}\times(2x - 5) + \frac{1}{x^3 + 7x + 9}\times\left(3x^2 + 7\right)\right] \\ &\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right] \\ &\Rightarrow \frac{dy}{dx} = \left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)\left[\frac{(2x - 5)\left(x^3 + 7x + 9\right) + \left(3x^2 + 7\right)\left(x^2 - 5x + 8\right)}{\left(x^2 - 5x + 8\right)\left(x^3 + 7x + 9\right)}\right] \\ &\Rightarrow \frac{dy}{dx} = 2x\left(x^3 + 7x + 9\right) - 5\left(x^3 + 7x + 9\right) + 3x^2\left(x^2 - 5x + 8\right) + 7\left(x^2 - 5x + 8\right) \\ &\Rightarrow \frac{dy}{dx} = \left(2x^4 + 14x^2 + 18x\right) - 5x^3 - 35x - 45 + \left(3x^4 - 15x^3 + 24x^2\right) + \left(7x^2 - 35x + 56\right) \\ &\Rightarrow \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \end{aligned}$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Answer needs Correction? **Click Here**

Q18 : If *u*, *v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer :

Let $y = u.v.w = u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v \cdot w)$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v \cdot w + u\left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y = u.v.w$, we obtain

$$\log y = \log u + \log v + \log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w.\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

Answer needs Correction? **Click Here**

Exercise 5.6 : Solutions of Questions on Page Number : 181

Q1 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2,\ y = at^4$

Answer :

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Answer needs Correction? Click Here

Q2 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\cos\theta,\ y = b\cos\theta$

Answer :

The given equations are $x = a\cos\theta$ and $y = b\cos\theta$

$$\text{Then, } \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Answer needs Correction? Click Here

Q3 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \sin t,\ y = \cos 2t$

Answer :

The given equations are $x = \sin t$ and $y = \cos 2t$

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t \cdot \frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2 \cdot 2\sin t \cos t}{\cos t} = -4\sin t$$

Answer needs Correction? **Click Here**

Q4 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 4t,\ y = \frac{4}{t}$

Answer :

The given equations are $x = 4t$ and $y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Answer needs Correction? **Click Here**

Q5 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$

Answer :

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

Then, $\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta - \cos 2\theta) = \frac{d}{d\theta}(\cos\theta) - \frac{d}{d\theta}(\cos 2\theta)$

$$= -\sin\theta - (-2\sin 2\theta) = 2\sin 2\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta) = \frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos\theta - 2\cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Answer needs Correction? **Click Here**

Q6 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

Answer :

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a[0 + (-\sin\theta)] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Answer needs Correction? **Click Here**

Q7 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Answer :

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}(\sin^3 t) - \sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t) - \sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t\cdot\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t\cdot\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\cos^2 t(-\sin t)-\cos^3 t\cdot\frac{1}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t\cdot\sqrt{\cos 2t}}$$

$$\therefore\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t(2\sin t\cos t)}{3\cos 2t\sin^2 t\cos t+\sin^3 t(2\sin t\cos t)}$$

$$=\frac{\sin t\cos t\left[-3\cos 2t.\cos t+2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t+2\sin^3 t\right]}$$

$$=\frac{\left[-3\left(2\cos^2 t-1\right)\cos t+2\cos^3 t\right]}{\left[3\left(1-2\sin^2 t\right)\sin t+2\sin^3 t\right]}\qquad\left[\begin{array}{l}\cos 2t=\left(2\cos^2 t-1\right),\\ \cos 2t=\left(1-2\sin^2 t\right)\end{array}\right]$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t}$$

$$=\frac{-\cos 3t}{\sin 3t}\qquad\left[\begin{array}{l}\cos 3t=4\cos^3 t-3\cos t,\\ \sin 3t=3\sin t-4\sin^3 t\end{array}\right]$$

$$=-\cot 3t$$

Answer needs Correction? **Click Here**

Q8 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$$x=a\left(\cos t+\log\tan\frac{t}{2}\right),\ y=a\sin t$$

Answer :

The given equations are $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$ and $y=a\sin t$

$$\text{Then, }\frac{dx}{dt}=a\cdot\left[\frac{d}{dt}(\cos t)+\frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot\frac{t}{2}\cdot\sec^2\frac{t}{2}\cdot\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a\frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t) = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

Answer needs Correction? **Click Here**

Q9 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a\sec\theta,\ y = b\tan\theta$

Answer :

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

Then, $\frac{dx}{d\theta} = a\cdot\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$

$$\frac{dy}{d\theta} = b\cdot\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a}\times\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Answer needs Correction? **Click Here**

Q10 : If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer :

The given equations are $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Then, $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$

$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$

$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right\}\right]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Answer needs Correction? **Click Here**

Q11 : If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer :

The given equations are $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$\Rightarrow x = \left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}$ and $y = \left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$

$\Rightarrow x = a^{\frac{1}{2}\sin^{-1}t}$ and $y = a^{\frac{1}{2}\cos^{-1}t}$

Consider $x = a^{\frac{1}{2}\sin^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log x = \frac{1}{2}\sin^{-1}t\log a$

$\therefore \frac{1}{x}\cdot\frac{dx}{dt} = \frac{1}{2}\log a\cdot\frac{d}{dt}\left(\sin^{-1}t\right)$

$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}}$

$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2\sqrt{1-t^2}}$

Then, consider $y = a^{\frac{1}{2}\cos^{-1}t}$

Taking logarithm on both the sides, we obtain

$\log y = \frac{1}{2}\cos^{-1}t\log a$

$\therefore \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\log a\cdot\frac{d}{dt}\left(\cos^{-1}t\right)$

$\Rightarrow \frac{dy}{dt} = \frac{y\log a}{2}\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$

$\Rightarrow \frac{dy}{dt} = \frac{-y\log a}{}$

$dt \quad 2\sqrt{1-t^2}$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}.$$

Hence, proved.

Answer needs Correction? **Click Here**

Exercise 5.7 : Solutions of Questions on Page Number : 183

Q1 : Find the second order derivatives of the function.

x^2+3x+2

Answer :

Let $y=x^2+3x+2$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

Answer needs Correction? Click Here

Q2 : Find the second order derivatives of the function.

x^{20}

Answer :

Let $y=x^{20}$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{20}\right)=20x^{19}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(20x^{19}\right)=20\frac{d}{dx}\left(x^{19}\right)=20\cdot 19\cdot x^{18}=380x^{18}$$

Answer needs Correction? Click Here

Q3 : Find the second order derivatives of the function.

$x\cdot\cos x$

Answer :

Let $y=x\cdot\cos x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot\cos x)=\cos x\cdot\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x\cdot 1+x(-\sin x)=\cos x-x\sin x$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}[\cos x-x\sin x]=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$

$$=-\sin x-\left[\sin x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\sin x)\right]$$

$$=-\sin x-(\sin x+x\cos x)$$

$= -(x\cos x + 2\sin x)$

Answer needs Correction? Click Here

Q4 : Find the second order derivatives of the function.

$\log x$

Answer :

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Answer needs Correction? Click Here

Q5 : Find the second order derivatives of the function.

$x^3 \log x$

Answer :

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x \cdot \frac{d}{dx}(x^3) + x^3 \cdot \frac{d}{dx}(\log x)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2(1 + 3\log x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2(1+3\log x)\right]$$

$$= (1+3\log x)\cdot\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(1+3\log x)$$

$$= (1+3\log x)\cdot 2x + x^2 \cdot \frac{3}{x}$$

$$= 2x + 6x\log x + 3x$$

$$= 5x + 6x\log x$$

$$= x(5 + 6\log x)$$

Answer needs Correction? Click Here

Q6 : Find the second order derivatives of the function.

$e^x \sin 5x$

Answer :

Let $y = e^x \sin 5x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right) = \sin 5x \cdot \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}\left(\sin 5x\right)$$
$$= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x) = e^x \sin 5x + e^x \cos 5x \cdot 5$$
$$= e^x\left(\sin 5x + 5\cos 5x\right)$$

Then, $$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^x\left(\sin 5x + 5\cos 5x\right)\right]$$
$$= \left(\sin 5x + 5\cos 5x\right) \cdot \frac{d}{dx}\left(e^x\right) + e^x \cdot \frac{d}{dx}\left(\sin 5x + 5\cos 5x\right)$$
$$= \left(\sin 5x + 5\cos 5x\right)e^x + e^x\left[\cos 5x \cdot \frac{d}{dx}(5x) + 5\left(-\sin 5x\right) \cdot \frac{d}{dx}(5x)\right]$$
$$= e^x\left(\sin 5x + 5\cos 5x\right) + e^x\left(5\cos 5x - 25\sin 5x\right)$$
$$= e^x\left(10\cos 5x - 24\sin 5x\right) = 2e^x\left(5\cos 5x - 12\sin 5x\right)$$

Answer needs Correction? Click Here

Q7 : Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer :

Let $y = e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{6x} \cdot \cos 3x\right) = \cos 3x \cdot \frac{d}{dx}\left(e^{6x}\right) + e^{6x} \cdot \frac{d}{dx}\left(\cos 3x\right)$$
$$= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx}(6x) + e^{6x} \cdot \left(-\sin 3x\right) \cdot \frac{d}{dx}(3x)$$
$$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \qquad \ldots(1)$$
$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right) = 6 \cdot \frac{d}{dx}\left(e^{6x}\cos 3x\right) - 3 \cdot \frac{d}{dx}\left(e^{6x}\sin 3x\right)$$
$$= 6 \cdot \left[6e^{6x}\cos 3x - 3e^{6x}\sin 3x\right] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx}\left(e^{6x}\right) + e^{6x} \cdot \frac{d}{dx}\left(\sin 3x\right)\right] \qquad [\text{Using } (1)]$$
$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3\left[\sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot \cos 3x \cdot 3\right]$$
$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$
$$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$$
$$= 9e^{6x}\left(3\cos 3x - 4\sin 3x\right)$$

Answer needs Correction? Click Here

Q8 : Find the second order derivatives of the function.

$\tan^{-1} x$

Answer :

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(1+x^2\right)^{-1} = (-1)\cdot\left(1+x^2\right)^{-2}\cdot\frac{d}{dx}\left(1+x^2\right)$$

$$= \frac{-1}{\left(1+x^2\right)^2}\times 2x = \frac{-2x}{\left(1+x^2\right)^2}$$

Answer needs Correction? Click Here

Q9 : Find the second order derivatives of the function.

$\log(\log x)$

Answer :

Let $y = \log(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\log x)\right] = \frac{1}{\log x}\cdot\frac{d}{dx}(\log x) = \frac{1}{x\log x} = (x\log x)^{-1}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[(x\log x)^{-1}\right] = (-1)\cdot(x\log x)^{-2}\cdot\frac{d}{dx}(x\log x)$$

$$= \frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log x)\right]$$

$$= \frac{-1}{(x\log x)^2}\cdot\left[\log x\cdot 1 + x\cdot\frac{1}{x}\right] = \frac{-(1+\log x)}{(x\log x)^2}$$

Answer needs Correction? Click Here

Q10 : Find the second order derivatives of the function.

$\sin(\log x)$

Answer :

Let $y = \sin(\log x)$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\log x)\right] = \cos(\log x)\cdot\frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$= \frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right] - \cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x) \right] - \cos(\log x).1}{x^2}$$

$$= \frac{-x \sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-\left[\sin(\log x) + \cos(\log x) \right]}{x^2}$$

Answer needs Correction? Click Here

Q11 : If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer :

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x) \right]$$

$$= -\left[5 \cdot \frac{d}{dx}(\sin x) + 3 \cdot \frac{d}{dx}(\cos x) \right]$$

$$= -\left[5\cos x + 3(-\sin x) \right]$$

$$= -\left[5\cos x - 3\sin x \right]$$

$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved.

Answer needs Correction? Click Here

Q12 : If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer :

It is given that, $y = \cos^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(1-x^2)^{\frac{-1}{2}} \right]$$

$$= -\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{-\frac{3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$= \frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{\left(1-x^2\right)^3}} \qquad \ldots(\text{i})$$

$y = \cos^{-1}x \Rightarrow x = \cos y$

Putting $x = \cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cot y\cdot\operatorname{cosec}^2 y$$

Answer needs Correction? **Click Here**

Q13 : If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2y_2 + xy_1 + y = 0$

Answer :

It is given that, $y = 3\cos(\log x) + 4\sin(\log x)$

Then,

$$y_1 = 3\cdot\frac{d}{dx}\left[\cos(\log x)\right] + 4\cdot\frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3\cdot\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] + 4\cdot\left[\cos(\log x)\cdot\frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\left\{4\cos(\log x) - 3\sin(\log x)\right\}' - \left\{4\cos(\log x) - 3\sin(\log x)\right\}(x)'}{x^2}$$

$$= \frac{x\left[4\left\{\cos(\log x)\right\}' - 3\left\{\sin(\log x)\right\}'\right] - \left\{4\cos(\log x) - 3\sin(\log x)\right\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= -4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)$$

$$-\frac{\quad}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Answer needs Correction? Click Here

Q14 : If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer :

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}\left(e^{mx}\right) + B \cdot \frac{d}{dx}\left(e^{nx}\right) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am \cdot \frac{d}{dx}\left(e^{mx}\right) + Bn \cdot \frac{d}{dx}\left(e^{nx}\right)$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2 e^{mx} + Bn^2 e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2 e^{mx} + Bn^2 e^{nx} - (m+n) \cdot \left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2 e^{mx} + Bn^2 e^{nx} - Am^2 e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2 e^{nx} + Amn e^{mx} + Bmn e^{nx}$$

$$= 0$$

Hence, proved.

Answer needs Correction? Click Here

Q15 : If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer :

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500 \cdot \frac{d}{dx}\left(e^{7x}\right) + 600 \cdot \frac{d}{dx}\left(e^{-7x}\right)$$

$$= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$

$$= 3500e^{7x} - 4200e^{-7x}$$

$$\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 3500 \cdot e^{7x}.\frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}$$
$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$
$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$
$$= 49y$$

Hence, proved.

Answer needs Correction? **Click Here**

Q16 : If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Answer :

The given relationship is $e^y(x+1) = 1$

$$e^y(x+1) = 1$$
$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y = \log\frac{1}{(x+1)}$$

Differentiating this relationship with respect to *x*, we obtain

$$\frac{dy}{dx} = (x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right) = (x+1).\frac{-1}{(x+1)^2} = \frac{-1}{x+1}$$
$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{1}{x+1}\right) = -\left(\frac{-1}{(x+1)^2}\right) = \frac{1}{(x+1)^2}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{-1}{x+1}\right)^2$$
$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Answer needs Correction? **Click Here**

Q17 : If $y = \left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2 + 2x\left(x^2+1\right)y_1 = 2$

Answer :

The given relationship is $y = \left(\tan^{-1}x\right)^2$

Then,

$$y_1 = 2\tan^{-1}x\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\Rightarrow y_1 = 2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \left(1+x^2\right)y_1 = 2\tan^{-1}x$$

Again differentiating with respect to x on both the sides, we obtain

$$\left(1+x^2\right)y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right)y_1 = 2$$

Hence, proved.

Answer needs Correction? **Click Here**

Exercise 5.8 : Solutions of Questions on Page Number : **186**

Q1 : Verify Rolle's Theorem for the function $f(x)=x^2+2x-8, x\in[-4,2]$

Answer :

The given function, $f(x)=x^2+2x-8$, being a polynomial function, is continuous in [- 4, 2] and is differentiable in (- 4, 2).

$$f(-4)=(-4)^2+2\times(-4)-8=16-8-8=0$$
$$f(2)=(2)^2+2\times2-8=4+4-8=0$$

$\therefore f(-4)=f(2)=0$

$\Rightarrow$ The value of $f(x)$ at - 4 and 2 coincides.

Rolle's Theorem states that there is a point $c\in(-4, 2)$ such that $f'(c)=0$

$$f(x)=x^2+2x-8$$
$$\Rightarrow f'(x)=2x+2$$
$$\therefore f'(c)=0$$
$$\Rightarrow 2c+2=0$$
$$\Rightarrow c=-1, \text{ where } c=-1\in(-4,2)$$

Hence, Rolle's Theorem is verified for the given function.

Answer needs Correction? **Click Here**

Q2 : Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5, 9]$

(ii) $f(x)=[x]$ for $x\in[-2, 2]$

(iii) $f(x)=x^2-1$ for $x\in[1, 2]$

Answer :

By Rolle's Theorem, for a function $f:[a, b]\to\mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

(c) $f(a)=f(b)$

then, there exists some $c\in(a, b)$ such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in [5, 9].

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in [- 2, 2].

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in (- 2, 2) is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (- 2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1) = (1)^2 - 1 = 0$

$f(2) = (2)^2 - 1 = 3$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Answer needs Correction? **Click Here**

Q3 : If $f:[-5,5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Answer :

It is given that $f:[-5,5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on [- 5, 5].

(b) f is differentiable on (- 5, 5).

Therefore, by the Mean Value Theorem, there exists $c \in$ (- 5, 5) such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$
$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$
$$\Rightarrow 10f'(c) \neq 0$$
$$\Rightarrow f(5) - f(-5) \neq 0$$
$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Answer needs Correction? **Click Here**

Q4 : Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer :

The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is $2x$ - 4.

$$f(1) = 1^2 - 4 \times 1 - 3 = -6,\ f(4) = 4^2 - 4 \times 4 - 3 = -3$$
$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in$ (1, 4) such that $f'(c) = 1$

$$f'(c) = 1$$
$$\Rightarrow 2c - 4 = 1$$
$$\Rightarrow c = \frac{5}{2} \text{ where } c = \frac{5}{2} \in (1, 4)$$

$\Rightarrow c = \frac{}{2}$, where $c = \frac{}{2} \in (1, 4)$

Hence, Mean Value Theorem is verified for the given function.

Answer needs Correction? **Click Here**

Q5 : Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval [*a*, *b*], where *a* = 1 and *b* = 3. Find all $c \in (1,3)$ for which $f'(c) = 0$

Answer :

The given function *f* is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1) = 1^3 - 5\times1^2 - 3\times1 = -7,\ f(3) = 3^3 - 5\times3^2 - 3\times3 = -27$$
$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{-27-(-7)}{3-1} = -10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$$f'(c) = -10$$
$$\Rightarrow 3c^2 - 10c - 3 = 10$$
$$\Rightarrow 3c^2 - 10c + 7 = 0$$
$$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$$
$$\Rightarrow 3c(c-1) - 7(c-1) = 0$$
$$\Rightarrow (c-1)(3c-7) = 0$$
$$\Rightarrow c = 1,\ \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1,\ 3)$ is the only point for which $f'(c) = 0$

Answer needs Correction? **Click Here**

Q6 : Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer :

Mean Value Theorem states that for a function $f : [a,\ b] \to \mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

then, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow$ $f(x)$ is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow$ $f(x)$ is not continuous in [- 2, 2].

The differentiability of f in (- 2, 2) is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (- 2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

It can be proved as follows.

$f(1)=1^2-1=0,\ f(2)=2^2-1=3$

$f(b)-f(a)\quad f(2)-f(1)\quad 3-0$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1} = \frac{3}{1} = 3$$

$$f'(x) = 2x$$

$$\therefore f'(c) = 3$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]$$

Exercise Miscellaneous : Solutions of Questions on Page Number : 191

Q1 : $(3x^2-9x+5)^9$

Answer :

Let $y=(3x^2-9x+5)^9$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}(3x^2-9x+5)^9$$
$$=9(3x^2-9x+5)^8\cdot\frac{d}{dx}(3x^2-9x+5)$$
$$=9(3x^2-9x+5)^8\cdot(6x-9)$$
$$=9(3x^2-9x+5)^8\cdot 3(2x-3)$$
$$=27(3x^2-9x+5)^8(2x-3)$$

Answer needs Correction? Click Here

Q2 : $\sin^3 x+\cos^6 x$

Answer :

Let $y=\sin^3 x+\cos^6 x$

$$\therefore\frac{dy}{dx}=\frac{d}{dx}(\sin^3 x)+\frac{d}{dx}(\cos^6 x)$$
$$=3\sin^2 x\cdot\frac{d}{dx}(\sin x)+6\cos^5 x\cdot\frac{d}{dx}(\cos x)$$
$$=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot(-\sin x)$$
$$=3\sin x\cos x(\sin x-2\cos^4 x)$$

Answer needs Correction? Click Here

Q3 : $(5x)^{3\cos 2x}$

Answer :

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y=3\cos 2x\log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=3\left[\log 5x\cdot\frac{d}{dx}(\cos 2x)+\cos 2x\cdot\frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x) + \cos 2x\cdot\frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$
$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x\log 5x + \frac{\cos 2x}{x}\right]$$
$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$$
$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$$

Answer needs Correction? **Click Here**

Q4 : $\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

Answer :

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$
$$= \frac{1}{\sqrt{1-x^3}}\times\frac{3}{2}\cdot x^{\frac{1}{2}}$$
$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Answer needs Correction? **Click Here**

Q5 : $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2 < x < 2$

Answer :

Let $y = \dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\,\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$= \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$= \frac{\sqrt{2x+7}\,\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Answer needs Correction? Click Here

Q6 : $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{}{2}$

Answer :

Let $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(1)

Then, $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$= \frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$= \frac{1+\cos x}{\sin x}$$

$$= \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y = \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = \frac{1}{}$$

$\Rightarrow \frac{dy}{dx} = \frac{\ }{2}$

Answer needs Correction? Click Here

Q7 : $(\log x)^{\log x}$, $x > 1$

Answer :

Let $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$\log y = \log x \cdot \log(\log x)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Answer needs Correction? Click Here

Q8 : $\cos(a\cos x + b\sin x)$, for some constant *a* and *b*.

Answer :

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \cdot \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \cdot \left[a(-\sin x) + b\cos x\right]$$

$$= (a\sin x - b\cos x) \cdot \sin(a\cos x + b\sin x)$$

Answer needs Correction? Click Here

Q9 : $(\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer :

Let $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$
$$\Rightarrow \log y=(\sin x-\cos x)\cdot\log(\sin x-\cos x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\cdot\frac{d}{dx}\log(\sin x-\cos x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot(\cos x+\sin x)+(\sin x-\cos x)\cdot\frac{1}{(\sin x-\cos x)}\cdot\frac{d}{dx}(\sin x-\cos x)$$
$$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x)\cdot\log(\sin x-\cos x)+(\cos x+\sin x)\right]$$
$$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

Answer needs Correction? **Click Here**

Q10 : $x^x+x^a+a^x+a^a$, for some fixed $a>0$ and $x>0$

Answer :

Let $y=x^x+x^a+a^x+a^a$

Also, let $x^x=u,\ x^a=v,\ a^x=w,$ and $a^a=s$

$\therefore y=u+v+w+s$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\frac{ds}{dx} \qquad ...(1)$$

$u=x^x$

$\Rightarrow \log u=\log x^x$

$\Rightarrow \log u=x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx}=\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx}=u\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx}=x^x\left[\log x+1\right]=x^x(1+\log x) \qquad ...(2)$$

$v=x^a$

$$\therefore \frac{dv}{dx}=\frac{d}{dx}\left(x^a\right)$$
$$\Rightarrow \frac{dv}{dx}=ax^{a-1} \qquad ...(3)$$

$w=a^x$

$\Rightarrow \log w=\log a^x$

$\Rightarrow \log w = x \log a$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$
$$\Rightarrow \frac{dw}{dx} = w \log a$$
$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad ...(4)$$

$s = a^a$

Since *a* is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \qquad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a + 0$$
$$= x^x(1+\log x) + ax^{a-1} + a^x \log a$$

Answer needs Correction? **Click Here**

Q11 : $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

Answer :

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\therefore y = u + v$

Differentiating both sides with respect to *x*,we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

$u = x^{x^2-3}$

$\therefore \log u = \log\left(x^{x^2-3}\right)$

$\log u = (x^2-3)\log x$

Differentiating with respect to *x*, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{d}{dx}(x^2-3) + (x^2-3) \cdot \frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \cdot 2x + (x^2-3) \cdot \frac{1}{x}$$
$$\Rightarrow \frac{du}{dx} = x^{x^2-3} \cdot \left[\frac{x^2-3}{x} + 2x \log x\right]$$

Also,

$v = (x-3)^{x^2}$

$\therefore \log v = \log(x-3)$

$\Rightarrow \log v = x^2 \log(x-3)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx} = \log(x-3)\cdot\frac{d}{dx}(x^2) + x^2\cdot\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \log(x-3)\cdot 2x + x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v\left[2x\log(x-3) + \frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

Answer needs Correction? **Click Here**

Q12 : Find $\frac{dy}{dx}$, if $y = 12(1-\cos t), x = 10(t-\sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$

Answer :

It is given that, $y = 12(1-\cos t), x = 10(t-\sin t)$

$$\therefore \frac{dx}{dt} = \frac{d}{dt}[10(t-\sin t)] = 10\cdot\frac{d}{dt}(t-\sin t) = 10(1-\cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[12(1-\cos t)] = 12\cdot\frac{d}{dt}(1-\cos t) = 12\cdot[0-(-\sin t)] = 12\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{12\sin t}{10(1-\cos t)} = \frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$$

Answer needs Correction? **Click Here**

Q13 : Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}, \ -1 \le x \le 1$

Answer :

It is given that, $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}\left(1-x^2\right)$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}}(-2x)$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$
$$\therefore \frac{dy}{dx} = 0$$

Answer needs Correction? Click Here

Q14 : If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, - 1 < x<1, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Answer :

It is given that,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y) = y^2(1+x)$$
$$\Rightarrow x^2 + x^2y = y^2 + xy^2$$
$$\Rightarrow x^2 - y^2 = xy^2 - x^2y$$
$$\Rightarrow x^2 - y^2 = xy(y-x)$$
$$\Rightarrow (x+y)(x-y) = xy(y-x)$$
$$\therefore x + y = -xy$$
$$\Rightarrow (1+x)y = -x$$
$$\Rightarrow y = \frac{-x}{(1+x)}$$

Differentiating both sides with respect to x, we obtain

$$y = \frac{-x}{(1+x)}$$
$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence, proved.

Answer needs Correction? Click Here

Q15 : If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2}{}}$ is a constant independent of $a$and b.

$\frac{a\ y}{dx^2}$

Answer :

It is given that, $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad ...(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$=-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}}=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$=-c$, which is constant and is independent of a and b

Hence, proved.

Answer needs Correction? **Click Here**

Q16 : If $\cos y=x\cos(a+y)$, with $\cos a\neq\pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Answer :

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \quad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Answer needs Correction? Click Here

Q17 : If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer :

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a\cdot\frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t\cdot\frac{d}{dx}(t) + t\cdot\frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a\cdot\frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t\cdot\frac{d}{dt}(t) + t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$= a[\cos t - \{\cos t - t\sin t\}] = at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$

$= \sec^2 t \cdot \frac{1}{at\cos t}$ $\left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$

$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$

Answer needs Correction? Click Here

Q18 : If $f(x) = |x|^3$, show that $f''(x)$ exists for all real *x*, and find it.

Answer :

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real *x* and is given by,

$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$

Answer needs Correction? Click Here

Q19 : Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers *n*.

Answer :

To prove: $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$

$\therefore$ P(n) is true for $n = 1$

Let P(k) is true for some positive integer k.

That is, $P(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that P(k+ 1) is also true.

Consider $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$

$= x^k \frac{d}{dx}(x) + x \frac{d}{dx}(x^k)$ [By applying product rule]

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \quad \text{[By applying product rule]}$$
$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$
$$= x^k + kx^k$$
$$= (k+1) \cdot x^k$$
$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Answer needs Correction? Click Here

Q20 : Using the fact that sin (A + B) = sin Acos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer :

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}[\sin(A+B)] = \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B)$$
$$\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) = \cos B \cdot \frac{d}{dx}(\sin A) + \sin A \cdot \frac{d}{dx}(\cos B)$$
$$+ \sin B \cdot \frac{d}{dx}(\cos A) + \cos A \cdot \frac{d}{dx}(\sin B)$$
$$\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) = \cos B \cdot \cos A \frac{dA}{dx} + \sin A(-\sin B)\frac{dB}{dx}$$
$$+ \sin B(-\sin A) \cdot \frac{dA}{dx} + \cos A \cos B \frac{dB}{dx}$$
$$\Rightarrow \cos(A+B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A \cos B - \sin A \sin B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$
$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Answer needs Correction? Click Here

Q21 : Does there exist a function which is continuos everywhere but not differentiable at exactly two points? Justify your answer ?

Answer :

Answer needs Correction? Click Here

Q22 : If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer :

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

Then, $\frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer needs Correction? Click Here

Q23 : If $y=e^{a\cos^{-1}x}, -1\le x\le 1$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Answer :

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$\log\ y=a\cos^{-1}x\log e$

$\log\ y=a\cos^{-1}x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=a\times\frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2=\frac{a^2y^2}{1-x^2}$$

$$\Rightarrow(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)+(1-x^2)\times\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]=a^2\frac{d}{dx}(y^2)$$

$$\Rightarrow\left(\frac{dy}{dx}\right)^2(-2x)+(1-x^2)\times2\frac{dy}{dx}.\frac{d^2y}{dx^2}=a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x)+\left(1-x^2\right)\times 2\frac{dy}{dx}.\frac{d^2y}{dx^2}=a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx}+\left(1-x^2\right)\frac{d^2y}{dx^2}=a^2.y \qquad \left[\frac{dy}{dx}\neq 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Hence, proved.

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